Period doubling in coupled maps

Sang-Yoon Kim
Department of Physics, Kangwon National University, Chunchon, Kangwon-Do 200-701, Korea

Hyungtae Kook
Department of Physics, Kyungwon University, Sungnam, Kyungki-Do 461-701, Korea
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Using a renormalization method, we study the critical behavior of period doubling in two coupled one-dimensional (1D) maps. We find three kinds of fixed maps of the period-doubling renormalization operator. Each of the fixed maps has a common relevant eigenvalue associated with scaling of the nonlinearity parameter of the uncoupled 1D map. However, the relevant "coupling" eigenvalues associated with scaling of the coupling parameter vary depending on the kind of fixed maps. The values of relevant eigenvalues agree well with those of the parameter scaling factors obtained by a direct numerical method. The renormalization results of two coupled maps are also extended to many coupled maps with a global coupling, in which each map is coupled to all the other maps with equal strength.

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I. INTRODUCTION

Since the discovery of the universal period-doubling route to chaos in low-dimensional maps [1,2], efforts have been made in studies of coupled maps to attempt to generalize to higher-dimensional maps [3–10]. Here we study the critical behavior of period doubling in two coupled one-dimensional (1D) maps by a renormalization method and show that the renormalization results are also extended to the case of many coupled 1D maps with a global coupling.

The renormalization method has played a central role in the study of critical behavior of period doubling in low-dimensional maps; the fixed maps of the period-doubling renormalization operator for 1D maps [1] and two-dimensional (2D) area-preserving maps [2] have been found. In the case of four-dimensional volume-preserving maps (two coupled 2D area-preserving maps), three fixed maps of an approximate renormalization operator have been found [7]. On the other hand, only one fixed map associated with the critical behavior at the zero coupling point has been found in the case of two coupled [3] and many coupled 1D maps [5,6].

In a recent numerical study on two coupled maps [10], we found an infinite number of critical points of period doubling. In a linear coupling case in which the leading term of the coupling function is linear, an infinite number of critical line segments, together with the previously known zero coupling point [3], constitute the critical set (the set of critical points), whereas in the case of nonlinear coupling, in which its leading term is nonlinear, the critical set consists of the only one critical line segment. The critical behavior varies depending on the position on the critical set. We found two kinds of new critical behavior at each critical line segment in the linear coupling case and one kind of new critical behavior at interior points of the critical line in the nonlinear coupling case, in addition to the critical behavior at the zero coupling point.

These numerical results imply that there exist three kinds of fixed maps of the renormalization operator in the space of two coupled maps. The numerical results on the critical behavior of two coupled maps are briefly surveyed in Sec. II. In Secs. III and IV, using a "reduced" equation scheme, we find two kinds of new fixed maps associated with the new critical behavior as well as the fixed map associated with the critical behavior at the zero coupling point. All the three kinds of fixed maps have a common (noncoordinate) relevant eigenvalue whose modulus is larger than unity, associated with scaling of the nonlinearity parameter of the uncoupled map. However, the relevant "coupling" eigenvalues associated with scaling of the coupling parameter depend on the kind of fixed maps. The values of relevant eigenvalues agree well with those of the parameter scaling factors obtained using a numerical method. In Sec. V fixed maps of the above three kinds are derived in several cases of coupling functions. In Sec. VI we extend the renormalization results of two coupled maps to many coupled maps with a global coupling, in which each map is coupled to all the other maps with equal strength. Finally, a summary is given in Sec. VII.

II. CRITICAL BEHAVIOR OF TWO COUPLED MAPS

We consider a map T consisting of two identical 1D maps coupled symmetrically:

$$T:\ \begin{cases} x_{i+1}=F(x_i,y_i)=f(x_i)+g(x_i,y_i)\ , \\ y_{i+1}=F(y_i,x_i)=f(y_i)+g(y_i,x_i)\ , \end{cases} \tag{2.1}$$

where the subscript i denotes the discrete time, $f(x)$ is an uncoupled 1D map with a quadratic maximum, and $g(x,y)$ is a coupling function. Here the uncoupled 1D map f satisfies a normalization condition

$$f(0)=1\ , \tag{2.2}$$

and the coupling function g obeys a condition

$$g(x,x)=0 \text{ for any } x\ . \tag{2.3}$$

The map (2.1) is invariant under the exchange of coordinates $x \leftrightarrow y$. The set of points which are invariant under the exchange of coordinates forms a symmetry line $y=x$. If an orbit lies on the symmetry line, then it is called an "in-phase" orbit; otherwise it is called an "out-of-phase" orbit. Here we study only in-phase orbits ($x_i=y_i$ for all i), which can easily be found from the uncoupled 1D map, $x_{i+1}=f(x_i)$, because of the condition (2.3).

Stability of an in-phase orbit of period p is determined from the Jacobian matrix M of T^p, which is the p product of the Jacobian matrix DT of T along the orbit

$$M=\prod_{i=1}^{p} DT(x_i,x_i) = \prod_{i=1}^{p} \begin{bmatrix} f'(x_i)-G(x_i) & G(x_i) \\ G(x_i) & f'(x_i)-G(x_i) \end{bmatrix}, \tag{2.4}$$

where $f'(x)=df(x)/dx$ and $G(x)=\partial g(x,y)/\partial y|_{y=x}$; hereafter, $G(x)$ will be referred to as the "reduced" coupling function of $g(x,y)$. The eigenvalues of M, called the stability multipliers of the orbit, are

$$\lambda_1=\prod_{i=1}^{p} f'(x_i)\ ,\quad \lambda_2=\prod_{i=1}^{p} [f'(x_i)-2G(x_i)]\ . \tag{2.5}$$

Note that λ_1 is just that of the uncoupled 1D map and the coupling affects only λ_2. An in-phase orbit is stable only when the moduli of both multipliers are less than unity, i.e., $-1<\lambda_i<1$ for $i=1, 2$.

An in-phase orbit can lose its stability only either by period-doubling bifurcation or tangent bifurcation; Hopf bifurcation does not occur since its stability multipliers (2.5) are always real. Between them, the successive period-doubling bifurcations complete an infinite sequence. We call the in-phase orbit of period 2^n created by the nth period-doubling bifurcation the in-phase orbit of level n. When its first stability multiplier $\lambda_{1,n}$ passes through -1, the orbit loses its stability via in-phase period-doubling bifurcation, giving rise to the creation of a period-doubled in-phase orbit of level $n+1$. Stable regions of orbits can be drawn in the parameter space of the coupling strength parameter (c) and the nonlinearity parameter (A). A period-doubling bifurcation point on the A axis for the 1D map is extended in the c axis direction for the coupled map to form a horizontal in-phase period-doubling bifurcation line because λ_1 is independent of the coupling strength parameter. Such bifurcation lines converge to the accumulation line at which the value of the nonlinearity parameter is just that of the accumulation point A^* of the 1D map.

The critical set for the coupled map lies on the accumulation line, but the structure of the critical set depends on the nature of coupling functions. The simplest case occurs when the reduced coupling function is identical to zero. In this case there is no coupling effect on the critical behavior because $\lambda_2=\lambda_1$; the whole accumulation line becomes the critical line and the critical behavior at any point on the critical line is the same as that of the uncoupled 1D map. Therefore the coupling affects the critical behavior only when the reduced coupling function is not identical to zero. In the latter case the structure of the critical set depends on the leading term of the coupling function and the critical behavior varies depending on the position on the critical set [10].

A "bifurcation path" in the A-c parameter space is formed by following the parameters (A_n,c_n) at which the in-phase orbit of level n has some given stability multiplier values (λ_1,λ_2). The scaling behavior of the sequence $[(A_n,c_n),\ n=0,1,2,\dots]$ can be determined by the scaling matrix method. The 2×2 scaling matrix of level n, Γ_n, is defined as follows:

$$\begin{bmatrix} A_{n-1}-A_{n-2} \\ c_{n-1}-c_{n-2} \end{bmatrix} = \Gamma_n \begin{bmatrix} A_n-A_{n-1} \\ c_n-c_{n-1} \end{bmatrix}\ . \tag{2.6}$$

As $n\to\infty$ the eigenvalues of Γ_n converge to the limits γ_1 and γ_2, which are just the parameter scaling factors of the sequence. With these scaling factors the sequence converges to a critical point (A^*,c^*). At this critical point, the stability multipliers $\lambda_{1,n}$ and $\lambda_{2,n}$ of the in-phase orbit of level n converge to the critical stability multipliers λ_1^* and λ_2^*:

$$\lambda_1^*=\lim_{n\to\infty}\lambda_{1,n}\ ,\quad \lambda_2^*=\lim_{n\to\infty}\lambda_{2,n}\ . \tag{2.7}$$

Since λ_1 depends only on the nonlinearity parameter A, γ_1 and λ_1^* are always the same as the 1D map values at all critical points: $\gamma_1=\delta=4.6692\dots$ and $\lambda_1^*=-1.6011\dots$ [1]. However, γ_2 and λ_2^* depend on the position on the critical set.

We briefly summarize the numerical results on the scaling behavior associated with coupling. In a linear coupling case, the critical set consists of an infinite number of critical line segments and the zero coupling point. There are three kinds of scaling behavior dependent on the position on the critical set. At the zero coupling point, $\gamma_2=-2.5029\dots$ and $\lambda_2^*=\lambda_1^*$, and at both ends of each critical line segment, $\gamma_2=1.9999\dots$ and $\lambda_2^*=1.0000\dots$. However, there exists no scaling factor γ_2 associated with coupling at interior points of each critical line segments since $\lambda_2^*=0$. Therefore the critical behavior at the interior points is essentially the same as that in the uncoupled 1D map. In a nonlinear coupling case, the critical set consists of the only one critical line segment, one end of which is the zero coupling point. At both ends of the critical line segment, $\gamma_2=1.9999\dots$ and $\lambda_2^*=\lambda_1^*$, but γ_2 is nonexistent at the interior points since $\lambda_2^*=0$. For more details, see Ref. [10].

III. RENORMALIZATION OPERATORS AND REDUCED FIXED MAPS

The period-doubling renormalization operator $\mathcal{N}$ for a coupled map T is composed of squaring (T^2, i.e., composing with itself) and rescaling (B) operators:

$$\mathcal{N}(T)\equiv BT^2B^{-1}\ . \tag{3.1}$$

Since we consider only in-phase orbits, the rescaling operator B is

$$B=\begin{bmatrix}\alpha & 0\\ 0 & \alpha\end{bmatrix} . \tag{3.2}$$

By applying the renormalization operator $\mathcal{N}$ to the coupled map (2.1), we obtain a renormalized map T_1:

$$T_1: \begin{cases} x_{i+1}=F_1(x_i,y_i)=\alpha F\left[F\left[\frac{x_i}{\alpha},\frac{y_i}{\alpha}\right],F\left[\frac{y_i}{\alpha},\frac{x_i}{\alpha}\right]\right] ,\\ y_{i+1}=F_1(y_i,x_i) . \end{cases} \tag{3.3}$$

Like the initial function F, the renormalized function F_1 can be separated into two parts, the uncoupled part f_1 and the coupling part g_1:

$$F_1(x,y)=f_1(x)+g_1(x,y) . \tag{3.4}$$

The renormalized coupling function also obeys the condition (2.3), i.e., $g_1(x,x)=0$ for any x. Therefore the renormalized uncoupled function f_1 satisfies

$$f_1(x)=F_1(x,x)=\alpha f\left[f\left[\frac{x}{\alpha}\right]\right] . \tag{3.5}$$

The rescaling factor α is chosen to preserve the normalization condition $f_1(0)=1$, i.e.,

$$\alpha=\frac{1}{f(1)} . \tag{3.6}$$

Subtracting f_1 from F_1, we obtain the renormalized coupling function

$$\begin{aligned} g_1(x,y)=\alpha f\left[\left[\frac{x}{\alpha}\right]+g\left[\frac{x}{\alpha},\frac{y}{\alpha}\right]\right] &\\ +\alpha g\left[f\left[\frac{x}{\alpha}\right]+g\left[\frac{x}{\alpha},\frac{y}{\alpha}\right],f\left[\frac{y}{\alpha}\right]\right.&\\ \left.+g\left[\frac{y}{\alpha},\frac{x}{\alpha}\right]\right]-\alpha f\left[f\left[\frac{x}{\alpha}\right]\right] .& \end{aligned} \tag{3.7}$$

Then Eqs. (3.5) and (3.7) define a renormalization operator $\mathcal{R}$ of transforming a pair of functions (f,g) [3]:

$$\begin{bmatrix} f_1\\ g_1\end{bmatrix}=\mathcal{R}\begin{bmatrix} f\\ g\end{bmatrix} . \tag{3.8}$$

By successive iterations of $\mathcal{R}$, we obtain the following recurrence equation:

$$\begin{bmatrix} f_{n+1}\\ g_{n+1}\end{bmatrix}=\mathcal{R}\begin{bmatrix} f_n\\ g_n\end{bmatrix} , \tag{3.9}$$

where f_n (g_n) is the uncoupled (coupling) part in the n times renormalized function F_n under the renormalization transformation $\mathcal{N}$, and the rescaling factor α is $1/f_n(1)$. That is,

$$f_{n+1}(x)=F_{n+1}(x,x)=\alpha f_n\left[f_n\left[\frac{x}{\alpha}\right]\right] , \tag{3.10}$$

$$\begin{aligned} g_{n+1}(x,y)=-\alpha f_n\left[f_n\left[\frac{x}{\alpha}\right]\right] &\\ +\alpha f_n\left[f_n\left[\frac{x}{\alpha}\right]+g_n\left[\frac{x}{\alpha},\frac{y}{\alpha}\right]\right] &\\ +\alpha g_n\left[f_n\left[\frac{x}{\alpha}\right]+g_n\left[\frac{x}{\alpha},\frac{y}{\alpha}\right],f_n\left[\frac{y}{\alpha}\right]\right.&\\ \left.+g_n\left[\frac{y}{\alpha},\frac{x}{\alpha}\right]\right] .& \end{aligned} \tag{3.11}$$

A map T_c with the nonlinearity and coupling parameters set to their critical values is called a critical map:

$$T_c: \begin{cases} x_{i+1}=F_c(x_i,y_i)=f_c(x_i)+g_c(x_i,y_i) ,\\ y_{i+1}=F_c(y_i,x_i) . \end{cases} \tag{3.12}$$

A critical map is attracted to a fixed map T^* under iterations of the renormalization transformation $\mathcal{N}$:

$$T^*: \begin{cases} x_{i+1}=F^*(x_i,y_i)=f^*(x_i)+g^*(x_i,y_i) ,\\ y_{i+1}=F^*(y_i,x_i) . \end{cases} \tag{3.13}$$

Here (f^*,g^*) is a fixed point of the renormalization transformation $\mathcal{R}$ with $\alpha=1/f^*(1)$:

$$\begin{bmatrix} f^*\\ g^*\end{bmatrix}=\mathcal{R}\begin{bmatrix} f^*\\ g^*\end{bmatrix} . \tag{3.14}$$

This fixed point-equation can be solved row by row consecutively. Note that the equation for f^* is just the fixed-point equation in the 1D map case, which has been solved numerically [1]. Therefore only the equation for the coupling fixed function g^* is left to be solved. One trivial solution is $g^*(x,y)=0$. For this zero coupling fixed function the fixed map (3.13) consists of two uncoupled 1D fixed maps. This fixed map is therefore associated with the critical behavior at the zero coupling point [3].

The numerical results, as summarized in the preceding section, suggest that there exist additional fixed points of $\mathcal{R}$, associated with the new critical behavior at critical points except the zero coupling point. However, it is not easy to directly solve the fixed point equation for $g^*(x,y)$. We therefore introduce a tractable recurrence equation for a reduced coupling function $G(x)=\partial g(x,y)/\partial y|_{y=x}$. Differentiating the recurrence equation (3.11) for $g(x,y)$ with respect to y and setting $y=x$, we obtain a recurrence equation for $G(x)$:

$$\begin{aligned} G_{n+1}(x)=\left[f_n'\left[f_n\left[\frac{x}{\alpha}\right]\right]-2G_n\left[f_n\left[\frac{x}{\alpha}\right]\right]\right]G_n\left[\frac{x}{\alpha}\right] &\\ +G_n\left[f_n\left[\frac{x}{\alpha}\right]\right]f_n'\left[\frac{x}{\alpha}\right] .& \end{aligned} \tag{3.15}$$

Then Eqs. (3.10) and (3.15) define a reduced renormalization operator $\tilde{\mathcal{R}}$ of transforming a pair of functions (f,G):

$$\begin{bmatrix} f_{n+1} \\ G_{n+1} \end{bmatrix} = \tilde{\mathcal{R}} \begin{bmatrix} f_n \\ G_n \end{bmatrix} . \tag{3.16}$$

We look for fixed points (f^*, G^*) of $\tilde{\mathcal{R}}$, which satisfy

$$\begin{bmatrix} f^* \\ G^* \end{bmatrix} = \tilde{\mathcal{R}} \begin{bmatrix} f^* \\ G^* \end{bmatrix} . \tag{3.17}$$

Here f^* is just the 1D fixed function and G^* is the reduced coupling fixed function of g^*, i.e., $G^*(x) = \partial g^*(x,y)/\partial y|_{y=x}$, which satisfies

$$G^*(x) = \left[f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right] - 2G^*\left[f^*\left[\frac{x}{\alpha}\right]\right]\right] G^*\left[\frac{x}{\alpha}\right] + G^*\left[f^*\left[\frac{x}{\alpha}\right]\right] f^{*\prime}\left[\frac{x}{\alpha}\right] . \tag{3.18}$$

We find three solutions for G^*:

$$G^*(x) = 0 , \tag{3.19}$$

$$G^*(x) = \tfrac{1}{2} f^{*\prime}(x) , \tag{3.20}$$

$$G^*(x) = \tfrac{1}{2}[f^{*\prime}(x) - 1] . \tag{3.21}$$

The second and third solutions are associated with the new critical behavior, as will be seen in the next section, while the first solution corresponds to the reduced coupling fixed function of the zero coupling fixed function $g^*(x,y) = 0$.

IV. LINEARIZED OPERATORS AND RELEVANT EIGENVALUES

Once a fixed function $F^*(x,y)$, or equivalently a coupling fixed function $g^*(x,y)$, is determined, its eigenvalues are obtained by linearizing the renormalization transformation around the fixed function and solving the resultant eigenvalue problem. In general, it is required to know the fixed function to linearize the transformation around it. However, in this section, before $g^*(x,y)$ is derived, it is shown that the eigenvalues are possibly obtained using the reduced coupling fixed function $G^*(x)$ rather than $g^*(x,y)$.

Let us examine the evolution of a pair of functions $(f^*(x)+h(x), g^*(x,y)+\varphi(x,y))$ close to a fixed point (f^*, g^*) under $\mathcal{R}$; here the perturbation φ to g^* also obeys the condition $\varphi(x,x)=0$. Linearizing the renormalization transformation $\mathcal{R}$ at the fixed point (f^*, g^*), we obtain the recurrence equation for the evolution of a pair of infinitesimal perturbations (h,φ):

$$\begin{bmatrix} h_{n+1} \\ \varphi_{n+1} \end{bmatrix} = \mathcal{L} \begin{bmatrix} h_n \\ \varphi_n \end{bmatrix} = \begin{bmatrix} \mathcal{L}_1 & 0 \\ \mathcal{L}_3 & \mathcal{L}_2 \end{bmatrix} \begin{bmatrix} h_n \\ \varphi_n \end{bmatrix} , \tag{4.1}$$

where

$$h_{n+1}(x) = [\mathcal{L}_1 h_n](x) = \alpha f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right] h_n\left[\frac{x}{\alpha}\right] + \alpha h_n\left[f^*\left[\frac{x}{\alpha}\right]\right] , \tag{4.2}$$

$$\varphi_{n+1}(x,y) = [\mathcal{L}_2 \varphi_n](x,y) + [\mathcal{L}_3 h_n](x) , \tag{4.3}$$

$$[\mathcal{L}_2\varphi_n](x,y) = \alpha F_1^*\left[F^*\left[\frac{x}{\alpha},\frac{y}{\alpha}\right], F^*\left[\frac{y}{\alpha},\frac{x}{\alpha}\right]\right]\varphi_n\left[\frac{x}{\alpha},\frac{y}{\alpha}\right] + \alpha F_2^*\left[F^*\left[\frac{x}{\alpha},\frac{y}{\alpha}\right], F^*\left[\frac{y}{\alpha},\frac{x}{\alpha}\right]\right]\varphi_n\left[\frac{y}{\alpha},\frac{x}{\alpha}\right] + \alpha\varphi_n\left[F^*\left[\frac{x}{\alpha},\frac{y}{\alpha}\right], F^*\left[\frac{y}{\alpha},\frac{x}{\alpha}\right]\right] , \tag{4.4}$$

$$[\mathcal{L}_3 h_n](x) = \alpha F_1^*\left[F^*\left[\frac{x}{\alpha},\frac{y}{\alpha}\right], F^*\left[\frac{y}{\alpha},\frac{x}{\alpha}\right]\right] h_n\left[\frac{x}{\alpha}\right] + \alpha F_2^*\left[F^*\left[\frac{x}{\alpha},\frac{y}{\alpha}\right], F^*\left[\frac{y}{\alpha},\frac{x}{\alpha}\right]\right] h_n\left[\frac{y}{\alpha}\right] + \alpha h_n\left[F^*\left[\frac{x}{\alpha},\frac{y}{\alpha}\right]\right] - [\mathcal{L}_1 h_n](x) . \tag{4.5}$$

Here $F^*(x,y) = f^*(x) + g^*(x,y)$ and the subscript i $(i=1,2)$ of F^* denotes the partial derivative with respect to the ith argument. Note that, although h_n couples to both h_{n+1} and φ_{n+1}, φ_n couples only to φ_{n+1}. The reducibility of $\mathcal{L}$ into a semiblock form implies that to determine the eigenvalues of $\mathcal{L}$ it is sufficient to work independently in each of $h(x)$ subspace and $\varphi(x,y)$ subspace. That is, one can find eigenvalues of $\mathcal{L}_1$ and $\mathcal{L}_2$ separately and they give the whole spectrum of $\mathcal{L}$.

A pair of perturbations (h,φ) is called an eigenperturbation with eigenvalue λ if

$$\lambda \begin{bmatrix} h \\ \varphi \end{bmatrix} = \mathcal{L} \begin{bmatrix} h \\ \varphi \end{bmatrix} , \tag{4.6}$$

that is,

$$\lambda h(x) = [\mathcal{L}_1 h](x) , \tag{4.7}$$

$$\lambda \varphi(x,y) = [\mathcal{L}_2 \varphi](x,y) + [\mathcal{L}_3 h](x) . \tag{4.8}$$

We first solve Eq. (4.7) to find eigenvalues of $\mathcal{L}_1$. Note that this is just the eigenvalue equation in the 1D map case. It has been shown in Ref. [1] that the equation has only one (noncoordinate) relevant eigenvalue δ ($=4.6692\ldots$). The eigenfunction $h(x)$ with the eigenvalue δ has also been obtained numerically. However, note that although the eigenvalue δ of $\mathcal{L}_1$ is also an eigenvalue of $\mathcal{L}$, $(h,0)$ is not an eigenperturbation of $\mathcal{L}$ unless $\mathcal{L}_3$ is a null operator.

Next, we consider a perturbation of the form $(0,\varphi)$ having only the coupling part. In this case $(0,\varphi)$ is an eigenperturbation of $\mathcal{L}$, only if φ satisfies

$$\lambda\varphi=\mathcal{L}_2\varphi\ . \tag{4.9}$$

The eigenvalues associated with the coupling perturbations will be called the "coupling"eigenvalues.

It is not easy to directly solve the eigenvalue equation for φ. We therefore introduce a tractable recurrence equation for a reduced coupling perturbation $\Phi(x)=\partial\varphi(x,y)/\partial y|_{y=x}$. In the case of general perturbations (h,φ), differentiating the recurrence equation (4.3) with respect to y and setting $y=x$, we obtain a recurrence equation for Φ:

$$\Phi_{n+1}(x)=[\tilde{\mathcal{L}}_2\Phi_n](x)+[\tilde{\mathcal{L}}_3 h_n](x)\ , \tag{4.10}$$

$$[\tilde{\mathcal{L}}_2\Phi_n](x)=\left[f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]-2G^*\left[f^*\left[\frac{x}{\alpha}\right]\right]\right]\Phi_n\left[\frac{x}{\alpha}\right]+\left[f^{*\prime}\left[\frac{x}{\alpha}\right]-2G^*\left[\frac{x}{\alpha}\right]\right]\Phi_n\left[f^*\left[\frac{x}{\alpha}\right]\right]\ , \tag{4.11}$$

$$\begin{aligned}[\tilde{\mathcal{L}}_3 h_n](x)=&\left[f^{*\prime\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]G^*\left[\frac{x}{\alpha}\right]-2G^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]G^*\left[\frac{x}{\alpha}\right]+G^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]f^{*\prime}\left[\frac{x}{\alpha}\right]\right]h_n\left[\frac{x}{\alpha}\right]\\&+G^*\left[\frac{x}{\alpha}\right]h_n'\left[f^*\left[\frac{x}{\alpha}\right]\right]+G^*\left[f^*\left[\frac{x}{\alpha}\right]\right]h_n'\left[\frac{x}{\alpha}\right]\ .\end{aligned} \tag{4.12}$$

Then the recurrence equations (4.2) and (4.10) for h and Φ define a reduced linear operator $\tilde{\mathcal{L}}$ of transforming a pair of perturbations (h,Φ):

$$\begin{bmatrix}h_{n+1}\\ \Phi_{n+1}\end{bmatrix}=\tilde{\mathcal{L}}\begin{bmatrix}h_n\\ \Phi_n\end{bmatrix}=\begin{bmatrix}\mathcal{L}_1 & 0\\ \tilde{\mathcal{L}}_3 & \tilde{\mathcal{L}}_2\end{bmatrix}\begin{bmatrix}h_n\\ \Phi_n\end{bmatrix}\ . \tag{4.13}$$

This equation can also be obtained by linearizing the reduced renormalization operator $\tilde{\mathcal{R}}$ of Eq. (3.16) at its fixed point (f^*, G^*).

The reducibility of $\tilde{\mathcal{L}}$ into a semiblock form again lets us search for the reduced coupling eigenperturbations of the form $(0,\Phi)$, where $\Phi(x)$ satisfies

$$\begin{aligned}\lambda\Phi(x)&=[\tilde{\mathcal{L}}_2\Phi](x)\\&=\left[f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]-2G^*\left[f^*\left[\frac{x}{\alpha}\right]\right]\right]\Phi\left[\frac{x}{\alpha}\right]+\left[f^{*\prime}\left[\frac{x}{\alpha}\right]-2G^*\left[\frac{x}{\alpha}\right]\right]\Phi\left[f^*\left[\frac{x}{\alpha}\right]\right]\ .\end{aligned} \tag{4.14}$$

If λ is an eigenvalue of $\mathcal{L}_2$, then it is also an eigenvalue of $\tilde{\mathcal{L}}_2$ unless $\Phi(x)=0$.

In the case $G^*(x)=0$, Eq. (4.14) becomes

$$\begin{aligned}\lambda\Phi(x)=&f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]\Phi\left[\frac{x}{\alpha}\right]\\&+f^{*\prime}\left[\frac{x}{\alpha}\right]\Phi\left[f^*\left[\frac{x}{\alpha}\right]\right]\ .\end{aligned} \tag{4.15}$$

Kuznetsov introduced Eq. (4.15) to obtain coupling eigenvalues in the case of the zero coupling fixed function $g^*(x,y)=0$ [3]; Eq. (4.14) holds for all the three cases of G^* in Eqs. (3.19)–(3.21) and hence it can be regarded as a generalized version of Eq. (4.15). He obtained two relevant coupling eigenvalues α and 2. At $x=0$, Eq. (4.15) becomes

$$\lambda\Phi(0)=f^{*\prime}(1)\Phi(0)=\alpha\Phi(0)\ . \tag{4.16}$$

There are two cases, linear and nonlinear coupling cases; $\Phi(0)\neq 0$ in linear coupling and $\Phi(0)=0$ in nonlinear coupling case. That is, in the case of a linear coupling perturbation, dividing both sides of Eq. (4.16) by $\Phi(0)$, we obtain $\lambda=\alpha$. In the other case, it can easily be seen that $\Phi(x)=f^{*\prime}(x)$ is a solution of Eq. (4.15) with $\lambda=2$. These values agree well with the numerical values of the parameter scaling factor γ_2 at the zero coupling point [10]: in the linear coupling case $\gamma_2=-2.5029\ldots$ and in the nonlinear coupling case $\gamma_2=1.9999\ldots$. The case of $G^*(x)=0$ is met not only at the zero coupling point but also at the other end of the critical line segment in the nonlinear coupling case, as will be seen in Sec. V.

In addition to the relevant coupling eigenvalues, we also obtain the critical stability multipliers (2.7). The invariance of a fixed map T^* under the renormalization transformation $\mathcal{N}$ implies that, if T^* has a periodic point (x,y) with period 2^n, then $B^{-1}(x,y)$ is a periodic point of T^* with period 2^{n+1}. Since rescaling leaves the stability multipliers (2.5) unaffected, all in-phase orbits of period 2^n for $n=0,1,\ldots$ have the same stability multipliers λ_1^* and λ_2^*, which are just the critical stability multipliers (2.7). That is, the critical stability multipliers have the values of the stability multipliers of the fixed point of the fixed map T^*:

$$\lambda_1^* = f^{*'}(\hat{x}) \ , \quad \lambda_2^* = f^{*'}(\hat{x}) - 2G^*(\hat{x}) \ , \tag{4.17}$$

where $\hat{x}$ ($=0.5493\ldots$) is the fixed point of the 1D fixed map, i.e., $\hat{x} = f^*(\hat{x})$. Here λ_1^* ($=-1.6011\ldots$) is just the critical stability multiplier in the uncoupled 1D map case and λ_2^* depends on the reduced coupling fixed function G^*. In the case $G^*(x)=0$, λ_2^* becomes the same as λ_1^*.

As shown earlier, there exist two additional reduced coupling fixed functions [see Eqs. (3.20) and (3.21)]. We first consider the case $G^*(x)=\frac{1}{2}f^{*'}(x)$. Substituting G^* into Eq. (4.17), we obtain the second critical stability multiplier $\lambda_2^*=0$. Note also that the reduced linear operator $\tilde{\mathcal{L}}_2$ of Eq. (4.14) becomes a null operator because the right-hand side of the equation is identical to zero for $G^*(x)=\frac{1}{2}f^{*'}(x)$. Therefore there exists no relevant coupling eigenvalue and, consequently, this reduced fixed point (f^*, G^*) has only one (noncoordinate) relevant eigenvalue δ associated with scaling of the nonlinearity parameter, like the uncoupled 1D map case. In Ref. [10], we found that, at interior points of the critical line segments in the linear and nonlinear coupling cases, the second critical stability multiplier is zero and there exists no parameter scaling factor γ_2 associated with coupling, i.e., the critical maps at interior points exhibit essentially 1D behavior, like the case of the 2D Hénon map with constant Jacobian [12]. The reduced fixed function $G^*(x)=\frac{1}{2}f^{*'}(x)$ therefore governs the critical behavior inside the critical line segments in both of the coupling cases.

Second, we consider the case, $G^*(x)=\frac{1}{2}[f^{*'}(x)-1]$. In this case the reduced eigenvalue equation (4.14) becomes

$$\lambda\Phi(x) = \Phi\left[\frac{x}{\alpha}\right] + \Phi\left[f^*\left[\frac{x}{\alpha}\right]\right] . \tag{4.18}$$

There exists a relevant coupling eigenvalue $\lambda=2$ when $\Phi(x)$ is a nonzero constant function, i.e., $\Phi(x)=\epsilon$ (ϵ is a nonzero constant). The value of the coupling eigenvalue agrees well with the numerical value of the second parameter scaling factor ($\gamma_2=1.9999\ldots$) [10] at both ends of each critical line segment in the linear coupling case. Substituting G^* into Eq. (4.17), we obtain the second critical stability multiplier $\lambda_2^*=1$, which also agrees well with the numerical value ($\lambda_2^*=1.0000\ldots$) [10]. Therefore this reduced fixed function is associated with the critical behavior at both ends of each critical line segment in the linear coupling case.

V. FIXED MAPS FOR TWO COUPLED MAPS

In this section we obtain fixed maps T^* of the renormalization transformation $\mathcal{N}$ by repeated actions of $\mathcal{N}$ on critical maps T_c. The critical maps converge under iterations of $\mathcal{N}$ to fixed maps of three kinds. In terms of the reduced coupling function $G(x)$, these fixed maps reduce to the three cases (3.19)–(3.21), and thereby the association with critical behavior is made in a natural way, as already explained in the preceding section.

In the following, we choose $f(x)=1-Ax^2$ and $g(x,y)=c(y-x)(d+ex+fy)$ as the uncoupled 1D map and the coupling function, respectively. Here A and c are the nonlinearity and the coupling parameter, d, e, and f are arbitrary constants. A critical point will be denoted as (c^*, A^*); A^* is always of the same value as that of the 1D map, i.e., $A^*=1.401\,155\ldots$. The reduced coupling function of g is $G(x)=c[d+(e+f)x]$.

We consider separately the cases of a nonlinear coupling and a linear coupling. In terms of the reduced coupling function $G(x)$, each case corresponds to the case $G(0)=0$ (nonlinear) and $G(0)\neq 0$ (linear), respectively. The nonlinear coupling case is further classified depending on whether $G(x)$ is identical to zero or not.

A. Nonlinear coupling when $G(x)=0$

To consider this case we set $d=0$ and $e+f=0$, that is, we have $g(x,y)=c(y-x)(ex+fy)$ and $G(x)=0$. In this case all critical maps are represented by a single point $(f_c,0)$ in the space of (f,G); f_c is the 1D critical map, i.e., $f_c(x)=1-A^*x^2$. The pair of initial functions $(f_c,0)$ is attracted to the reduced fixed point $(f^*,0)$ under iterations of the reduced renormalization transformation $\tilde{\mathcal{R}}$ of Eq. (3.16). In the c-A plane, therefore, the $A=A^*$ line itself becomes the critical line. Inserting $f=-e$ in $g(x,y)$, the critical maps have the following form:

$$\begin{aligned} x_{i+1} &= F_c(x_i,y_i) = f_c(x) + g_c(x,y) \\ &= \frac{\sqrt{\mu}+1}{2\sqrt{\mu}} f_c\left[\frac{x_i+y_i}{2} + \sqrt{\mu}\frac{x_i-y_i}{2}\right] \\ &\quad + \frac{\sqrt{\mu}-1}{2\sqrt{\mu}} f_c\left[\frac{x_i+y_i}{2} - \sqrt{\mu}\frac{x_i-y_i}{2}\right] , \\ y_{i+1} &= F_c(y_i,x_i) , \end{aligned} \tag{5.1}$$

where $\mu=1+(4c/A^*)e$. It can easily be shown that under iterations of $\mathcal{N}$ the critical map is attracted to the fixed map:

$$\begin{aligned} x_{i+1} &= F^*(x_i,y_i) = f^*(x) + g^*(x,y) \\ &= \frac{\sqrt{\mu}+1}{2\sqrt{\mu}} f^*\left[\frac{x_i+y_i}{2} + \sqrt{\mu}\frac{x_i-y_i}{2}\right] \\ &\quad + \frac{\sqrt{\mu}-1}{2\sqrt{\mu}} f^*\left[\frac{x_i+y_i}{2} - \sqrt{\mu}\frac{x_i-y_i}{2}\right] , \end{aligned} \tag{5.2}$$

$$y_{i+1} = F^*(y_i,x_i) \ .$$

Therefore, when $G(x)$ is identical to zero, all critical maps are attracted to the fixed maps of the form (5.2), which is a one-parameter (μ) family of fixed maps. The critical behavior is, of course, the same as the one analyzed for the case $G^*(x)=0$ in the preceding section.

Let us now introduce new coordinates X and Y:

$$X = \frac{x+y}{2} \ , \quad Y = \frac{x-y}{2} \ . \tag{5.3}$$

Under the coordinate change, the symmetry line $y=x$ is transformed into $Y=0$. That is, in terms of the new coordinates, the coupled map (2.1) becomes invariant under the reflection operation $Y\to -Y$. However, the coordinate change (5.3) leaves the rescaling operator (3.2) unaffected.

In the new coordinates, the fixed map (5.2) becomes

$$X_{i+1}=\tfrac{1}{2}[f^*(X_i+\sqrt{\mu}Y_i)+f^*(X_i-\sqrt{\mu}Y_i)]\ ,$$
$$Y_{i+1}=\frac{1}{2\sqrt{\mu}}[f^*(X_i+\sqrt{\mu}Y_i)-f^*(X_i-\sqrt{\mu}Y_i)]\ . \tag{5.4}$$

Note that the normalization condition $f^*(0)=1$ fixes the scale of X, whereas the scale of Y has not been fixed yet. This explains why the critical maps converge to a one-parameter family of fixed maps (5.4). A compact form of the fixed maps may be obtained by a scale change of Y.

(1) $\mu>0$, scale change $Y\to Y/\sqrt{\mu}$:

$$X_{i+1}=\tfrac{1}{2}[f^*(X_i+Y_i)+f^*(X_i-Y_i)]\ ,$$
$$Y_{i+1}=\tfrac{1}{2}[f^*(X_i+Y_i)-f^*(X_i-Y_i)]\ . \tag{5.5}$$

(2) $\mu<0$, scale change $Y\to Y/\sqrt{-\mu}$:

$$X_{i+1}=\tfrac{1}{2}[f^*(X_i+jY_i)+f^*(X_i-jY_i)]\ ,$$
$$Y_{i+1}=\frac{1}{2j}[f^*(X_i+jY_i)-f^*(X_i-jY_i)]\ , \tag{5.6}$$

where $j\equiv\sqrt{-1}$.

(3) $\mu=0$, no scale change:

$$X_{i+1}=f^*(X_i)\ ,\quad Y_{i+1}=f^{*\prime}(X_i)Y_i\ . \tag{5.7}$$

Thus, when $G^*(x)=0$ we have the three "representative" fixed maps with $\mu=\pm 1,0$.

An infinitesimal scale change of Y, i.e., a transformation $X\to X$ and $Y\to(1+\epsilon)Y$, corresponds to the following change of the original coordinates x and y:

$$x\to x+\epsilon\frac{x-y}{2}\ ,\quad y\to y-\epsilon\frac{x-y}{2}\ . \tag{5.8}$$

It can easily be shown that the perturbation associated with the coordinate change (5.8) is the coupling eigenperturbation $(0,\varphi)$ of $\mathcal{L}$ with a marginal eigenvalue $(\lambda=1)$, where

$$\varphi(x,y)=F^*(y,x)-F^*(x,y)$$
$$+(x-y)[F_1^*(x,y)-F_2^*(y,x)]\ . \tag{5.9}$$

Here the subscript i $(i=1,2)$ of F^* denotes the partial derivative of F^* with respect to the ith argument. In Appendix A we also obtain the coupling eigenvalues α^{-n} $(n=1,2,\ldots)$ associated with other coordinate changes.

B. Nonlinear coupling when $G(x)\neq 0$

In this case we set $g(x,y)=c(y-x)(ex+fy)$, where $e+f\neq 0$. The reduced coupling function becomes $G(x)=\epsilon f_c'(x)$, where $\epsilon=-(c/2A^*)(e+f)$. By successive actions of $\tilde{\mathcal{R}}$ on (f_c,G), we obtain

$$f_n(x)=\alpha f_{n-1}\left[f_{n-1}\left[\frac{x}{\alpha}\right]\right]\ ,\quad G_n(x)=\epsilon_n f_n'(x)\ , \tag{5.10}$$

$$\epsilon_n=2\epsilon_{n-1}-2\epsilon_{n-1}^2\ , \tag{5.11}$$

where $f_0(x)=f_c(x)$, $G_0(x)=G(x)$, and $\epsilon_0=\epsilon$. Here f_n converges to the 1D fixed function $f^*(x)$.

The recurrence equation (5.11) for ϵ has two fixed points ϵ^*:

$$\epsilon^*=0,\frac{1}{2}\ . \tag{5.12}$$

Stability of the fixed point ϵ^* is determined by its stability multiplier λ, where $\lambda=d\epsilon_n/d\epsilon_{n-1}|_{\epsilon^*}$. The fixed point at $\epsilon^*=\frac{1}{2}$ is superstable $(\lambda=0)$, while the other at $\epsilon^*=0$ is unstable $(\lambda=2)$. The basin of attraction to the superstable fixed point is the open interval (0,1), that is, any initial ϵ inside the unit interval $0<\epsilon<1$ converges to $\epsilon^*=\frac{1}{2}$. The unstable fixed point at $(\epsilon^*=0)$ is also the image of the other boundary point at $\epsilon=1$ under the recurrence equation (5.11). All points outside the unit interval diverge to the minus infinity. Consequently, in this nonlinear coupling case the critical set is the line segment joining two end points $c_1^*(=0)$ and $c_2^*[=-2A^*/(e+f)]$ on the $A=A^*$ line; the two end points c_1^* and c_2^* correspond to $\epsilon=0$ and 1, respectively. Inside the critical line segment all critical maps are attracted to the fixed maps whose reduced coupling function is $G^*(x)=\frac{1}{2}f^{*\prime}(x)$, while the critical maps at both ends of the line segment are attracted to the fixed maps whose reduced coupling fixed function is $G^*(x)=0$, like the case when $G(x)$ is identical to zero.

At the zero coupling point c_1^*, the critical map has the form of Eq. (5.1) with $\mu=1$ and therefore it is attracted to the fixed map (5.2) with $\mu=1$. At the other end point c_2^*, the critical map T_c has the following form:

$$x_{i+1}=F_c(x_i,y_i)=f_c(x)+g_c(x,y)$$
$$=\frac{\sqrt{\mu}-1}{2\sqrt{\mu}}f_c\left[\frac{x_i+y_i}{2}+\sqrt{\mu}\frac{x_i-y_i}{2}\right]$$
$$+\frac{\sqrt{\mu}+1}{2\sqrt{\mu}}f_c\left[\frac{x_i+y_i}{2}-\sqrt{\mu}\frac{x_i-y_i}{2}\right]\ , \tag{5.13}$$
$$y_{i+1}=F_c(y_i,x_i)\ ,$$

where $\mu=(-3e+5f)/(e+f)$. Note the difference between the two critical maps of Eqs. (5.1) and (5.13). However, the renormalized map T_1 of T_c under $\mathcal{N}$ has the form of Eq. (5.1). Therefore, the critical map (5.13) is also attracted to the fixed map (5.2) with $\mu=(-3e+5f)/(e+f)$.

Inside the critical line segment, the critical map T_c is of the form

$$x_{i+1}=F_c(x_i,y_i)=f_c\left\{\left[\left[\frac{x_i+y_i}{2}\right]^2+\mu\left[\frac{x_i-y_i}{2}\right]^2+\nu(x_i^2-y_i^2)\right]^{1/2}\right\},$$
$$y_{i+1}=F_c(y_i,x_i)\ , \tag{5.14}$$

where $\mu=1+(2c/A^*)(e-f)$ and $\nu=\frac{1}{2}[1+(c/A^*)(e+f)]$; the range of ν inside the critical line segment is $-\frac{1}{2}<\nu<\frac{1}{2}$. It is easy to see that the critical map at the middle point ($\nu=0$, or $c=c_2^*/2$) is attracted to the fixed map of the form

$$x_{i+1}=f^*\left\{\left[\left[\frac{x_i+y_i}{2}\right]^2+\mu\left[\frac{x_i-y_i}{2}\right]^2\right]^{1/2}\right\},$$
$$y_{i+1}=F^*(y_i,x_i)\ , \tag{5.15}$$

where $\mu=(-e+3f)/(e+f)$.

In the system of coordinates X and Y, the fixed map (5.15) becomes

$$X_{i+1}=f^*[(X_i^2+\mu Y_i^2)^{1/2}]\ ,$$
$$Y_{i+1}=0\ . \tag{5.16}$$

Like the case $G^*(x)=0$, we have three representative fixed maps with $\mu=\pm1,0$ by a scale change in Y.

(1) $\mu>0$, scale change $Y\to Y/\sqrt{\mu}$:

$$X_{i+1}=f^*[(X_i^2+Y_i^2)^{1/2}]\ ,\quad Y_{i+1}=0\ . \tag{5.17}$$

(2) $\mu<0$, scale change $Y\to Y/\sqrt{-\mu}$:

$$X_{i+1}=f^*[(X_i^2-Y_i^2)^{1/2}]\ ,\quad Y_{i+1}=0\ . \tag{5.18}$$

(3) $\mu=0$, no scale change:

$$X_{i+1}=f^*(X_i)\ ,\quad Y_{i+1}=0\ . \tag{5.19}$$

Note that all images $\{(X_n,Y_n);\ n=1,2,\ldots\}$ of any initial point (X_0,Y_0) under these fixed maps lie on the symmetry line $Y=0$. Therefore, these fixed maps exhibit essentially 1D behavior.

At the other interior points ($\nu\neq0$), unfortunately we could not analytically apply $\mathcal{N}$ to the critical maps. By numerical implementation of the method of Appendix B, we determine fixed maps up to the quadratic terms (hereafter, these fixed maps will be called the quadratic fixed maps):

$$X_{i+1}=1+C_1(X_i^2+\mu Y_i^2)\ ,\quad Y_{i+1}=0\ , \tag{5.20}$$

where $C_1=-1.5276\ldots$, and $\mu\cong1+(2c/A^*)(e-f)$. Note that the value of C_1 agrees well with the numerical value of the coefficient of the quadratic term in the 1D fixed function [1]. Therefore it would be reasonable to believe that the quadratic fixed map (5.20) is a truncated one of the fixed map (5.16) at its quadratic terms. To be more convinced, the method used in Ref. [6] was also applied to numerically iterate $\mathcal{N}$. In the latter method the function value of the n times renormalized map is calculated at a given (x,y) point and compared with the function value of the fixed map (5.16) at the same point, which showed a fairly close coincidence at a number of points. In fact, to identify a function, comparing coefficients is evidently much more efficient and reliable than comparing function values.

When $\mu=\pm1$, the fixed map (5.16) can be further transformed to the form

$$X_{i+1}=f^*[(X_i^2-Y_i)^{1/2}]\ ,\quad Y_{i+1}=0 \tag{5.21}$$

by a coordinate change $X\to X$ and $Y\to\sqrt{\pm Y}$ (the signs $-$ and $+$ correspond to the cases $\mu=1$ and -1, respectively). The domain of Y in the map (5.21) is restricted to $Y\leq0$ for $\mu=1$ and $Y\geq0$ for $\mu=-1$ cases. Combining the two cases of $\mu=\pm1$, the domain of Y in the map becomes the whole real line. Note also that in the new coordinates the rescaling operator B becomes

$$B=\begin{bmatrix}\alpha & 0\\ 0 & \alpha^2\end{bmatrix}. \tag{5.22}$$

Obviously, the map (5.21) is the fixed map of the renormalization transformation with the rescaling operator (5.22), which was found in Ref. [13] where Collet, Eckmann, and Koch extended the universality of 1D maps to higher-dimensional dissipative maps.

C. Linear coupling case

We now consider a linear coupling case. As an example, we choose a linearly coupled map with $g(x,y)=c(y-x)$. In this case the numerical work [10] has shown that the critical set consists of the zero coupling point and an infinite set of critical line segments accumulating to the zero coupling point. There exist three kinds of critical behavior depending on the position on the critical set: the zero coupling point, two ends of each critical line segment, and the interior points of each critical line segment.

As already shown in Sec. V A, the critical map at the zero coupling point is attracted to the fixed map (5.2) with $\mu=1$. However, repeated actions of $\mathcal{N}$ on the other critical maps could not be done analytically. Instead, employing again the method of Appendix B, we numerically obtain quadratic fixed maps. For an example, consider the leftmost critical line segment joining two end points c_L ($=-1.457\,727\ldots$) and c_R ($=-1.013\,402\ldots$) on the $A=A^*$ line [11]. Critical maps at the interior points ($c_L<c<c_R$) are attracted to the fixed maps of the form (5.20), where $C_1=-1.5276\ldots$ and the value of μ varies depending on c (see Fig. 1). It is also believed that this quadratic fixed map indicates the fixed map (5.16) truncated at its quadratic terms.

At both ends ($c=c_L,c_R$), the quadratic fixed map is

$$X_{i+1}=1+C_1X_i^2\ ,\quad Y_{i+1}=Y_i\ , \tag{5.23}$$

where $C_1=-1.5276\ldots$. The form of Eq. (5.23) suggests that the corresponding fixed map be of the following form:

$$X_{i+1}=f^*(X_i)\ ,\quad Y_{i+1}=Y_i\ , \tag{5.24}$$

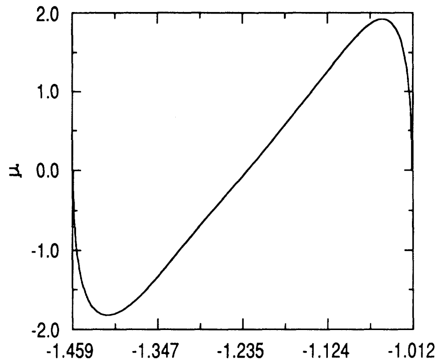

FIG. 1. Plot of μ for $c_L < c < c_R$.

which seems to be correct when the numerical method of Ref. [6] is applied to double-check. Obviously, the map (5.24) is invariant under $\mathcal{N}$, and thus we find the third kind of the fixed map. The fixed map (5.24) is invariant under a scale change of Y, unlike the above two kinds of the fixed maps (5.4) and (5.16).

Consider an infinitesimal perturbation $\epsilon\delta T$ to the fixed map (5.24):

$$\epsilon\delta T\colon X_{i+1}=0\ ,\quad Y_{i+1}=\epsilon Y_i\ . \tag{5.25}$$

Then the renormalized map T_1 of T $(=T^*+\epsilon\delta T)$ under $\mathcal{N}$ is

$$X_{i+1}=f^*(X_i)\ ,\quad Y_{i+1}=Y_i+2\epsilon Y_i+O(\epsilon^2)\ . \tag{5.26}$$

Therefore, the perturbation $\epsilon\delta T$ is an eigenperturbation with eigenvalue $\lambda=2$.

In the original coordinates x and y, the fixed map (5.24) is of the form

$$x_{i+1}=F^*(x_i,y_i)=f^*(x_i)+g^*(x_i,y_i) =f^*\left[\frac{x_i+y_i}{2}\right]+\frac{x_i-y_i}{2}\ , \tag{5.27}$$
$$y_{i+1}=F^*(y_i,x_i)\ .$$

In this case, the reduced coupling fixed function is $G^*(x)=\frac{1}{2}[f^{*\prime}(x)-1]$. Therefore, the fixed map (5.27) corresponds to the third solution (3.21) for G^*. As shown in the preceding paragraph, there exists a coupling eigenperturbation $(0,\varphi)$ with eigenvalue $\lambda=2$, where $\varphi(x,y)=x-y$. The reduced coupling eigenfunction of φ is $\Phi(x)=-1$, which is just the solution (with $\lambda=2$) of the reduced eigenvalue equation (4.18) in the case $G^*(x)=\frac{1}{2}[f^{*\prime}(x)-1]$.

VI. MANY COUPLED MAPS

Recently much attention has been paid to globally coupled dynamical systems of many elements, in which each element is coupled to all the other elements with equal strength [6,14–21]. This kind of globally coupled systems with continuous or discrete time describes diverse phenomena such as charge density waves [14], Josephson-junction arrays [15–18], and p-n junction arrays [19]. Such dynamical systems are known to exhibit period-doubling or tangent bifurcations [6,15], mode locking [14], attractor crowding [16], neutral stability of "antiphase" states [17,18], and clustering [19–21].

Here we study the critical behavior of period doubling of in-phase orbits in many coupled 1D maps with a global coupling, in which each 1D map is coupled to all the other 1D maps with equal strength. We show that the renormalization results of two coupled maps are straightforwardly extended to this kind of many coupled maps.

A. Many coupled maps with a global coupling

Consider an N-coupled map with a periodic boundary condition:

$$T\colon x_m(t+1)=F(\sigma^{m-1}\mathbf{x}(t)) =F(x_m(t),x_{m+1}(t),\dots,x_{m-1}(t))\ ,\quad m=1,\dots,N\ , \tag{6.1}$$

where N is a positive integer larger than or equal to 2, $x_m(t)$ is the state of the mth element at a discrete time t, $\mathbf{x}=(x_1,\dots,x_N)$, σ is the cyclic permutation of the elements of $\mathbf{x}$, i.e., $\sigma\mathbf{x}=(x_2,\dots,x_1)$, and σ^{m-1} means $(m-1)$ applications of σ. The periodic condition imposes $x_m(t)=x_{m+N}(t)$ for all m. Like the two-coupled map case $(N=2)$, the function F consists of two parts:

$$F(\mathbf{x})=f(x_1)+g(\mathbf{x})\ , \tag{6.2}$$

where f is an uncoupled 1D map with a quadratic maximum and g is a coupling function. Thus the map T of Eq. (6.1) becomes

$$T\colon x_m(t+1)=f(x_m(t)) +g(x_m(t),x_{m+1}(t),\dots,x_{m-1}(t))\ ,\quad m=1,\dots,N\ . \tag{6.3}$$

The uncoupled 1D map f satisfies the normalization condition

$$f(0)=1\ , \tag{6.4}$$

and the coupling function obeys the condition

$$g(x,\dots,x)=0 \text{ for any } x\ . \tag{6.5}$$

Here we study a globally coupled map with a coupling function g of the form

$$g(\mathbf{x})=\frac{J}{N}\sum_{i=1}^{N}[u(x_i)-u(x_1)]\ , \tag{6.6}$$

where $u(x)$ is a function of one variable. Since the value of g for $x_1=\cdots=x_N=x$ becomes zero, g satisfies the condition (6.5). In this global coupling case, each 1D map is coupled to all the other 1D maps with equal coupling strength J/N inversely proportional to the number of degrees of freedom N (hereafter, J will be called the coupling parameter).

This globally coupled map has the permutation symmetry, that is, it is invariant under exchange of any two elements $x_k \leftrightarrow x_l$. In the N-dimensional phase space of $(x_1, \ldots, x_N)$, the set of invariant points forms a symmetry line on which $x_1 = \cdots = x_N$. If an orbit lies on the symmetry line, then it is called the in-phase orbit; $x_1(t) = \cdots = x_N(t) = x(t)$, i.e., $\mathbf{x}(t) = (x(t), \ldots, x(t))$ for all t. Here we study only in-phase orbits.

B. Stability of orbits and the critical behavior

The stability analysis of an orbit in many coupled maps is conveniently carried out by Fourier transforming with respect to the discrete space $\{m\}$ [4,6]. Consider an orbit $\{\mathbf{x}(t)\} \equiv \{x_m(t);\ m=1, \ldots, N\}$. The discrete spatial Fourier transform of the orbit is

$$\mathcal{F}[x_m(t)] \equiv \frac{1}{N} \sum_{m=1}^{N} e^{-2\pi i m j/N} x_m(t) = \xi_j(t)\ , \quad j=0,1,\ldots,N-1\ . \tag{6.7}$$

The wavelength of the jth Fourier mode is N/j.

To determine stability of an in-phase orbit we consider an infinitesimal perturbation $\{\delta x_m(t)\}$ to the in-phase orbit, i.e., $x_m(t) = x(t) + \delta x_m(t)$ for $m=1, \ldots, N$; $\delta x_m(t)$'s are not necessarily of in-phase in general. Since $g(x, \ldots, x) = 0$, $x(t)$ is found from the uncoupled 1D map, i.e., $x(t+1) = f(x(t))$. Linearizing the N-coupled map (6.3) at the in-phase orbit, we obtain

$$\delta x_m(t+1) = f'(x(t))\delta x_m(t) + \sum_{l=1}^{N} G^{(l)}(x(t))\delta x_{l+m-1}(t)\ , \tag{6.8}$$

where

$$\begin{aligned} f'(x) &= \frac{df}{dx}\ , \\ G^{(l)}(x) &= \left.\frac{\partial g(\sigma^{(m-1)}\mathbf{x})}{\partial x_{l+m-1}}\right|_{x_1=\cdots=x_N=x} \\ &= \left.\frac{\partial g(\mathbf{x})}{\partial x_l}\right|_{x_1=\cdots=x_N=x}\ . \end{aligned} \tag{6.9}$$

Let $\delta\xi_j(t)$ be the Fourier transform of $\delta x_m(t)$, i.e.,

$$\delta\xi_j(t) = \mathcal{F}[\delta x_m(t)] = \frac{1}{N}\sum_{m=1}^{N} e^{-2\pi i m j/N}\delta x_m(t)\ , \quad j=0,1,\ldots,N-1\ . \tag{6.10}$$

Then the Fourier transform of Eq. (6.8) becomes

$$\delta\xi_j(t+1) = \left[f'(x(t)) + \sum_{l=1}^{N} G^{(l)}(x(t)) e^{2\pi i(l-1)j/N} \right] \delta\xi_j(t)\ , \quad j=0,1,\ldots,N-1\ . \tag{6.11}$$

For an in-phase orbit with period p, its linear stability is determined by iterating the linearized map (6.11) p times:

$$\delta\xi_j(t+p) = \prod_{n=0}^{p-1} \left[f'(x(t+n)) + \sum_{l=1}^{N} G^{(l)}(x(t+n)) e^{2\pi i(l-1)j/N} \right] \delta\xi_j(t)\ , \quad j=0,1,\ldots,N-1\ . \tag{6.12}$$

That is, the stability multipliers of the orbit are

$$\lambda_j = \prod_{t=0}^{p-1} \left[f'(x(t)) + \sum_{l=1}^{N} G^{(l)}(x(t)) e^{2\pi i(l-1)j/N} \right]\ , \quad j=0,1,\ldots,N-1\ . \tag{6.13}$$

Here each λ_j is associated with stability of the mode of wave number j. An in-phase orbit is stable only when all its modes are stable, i.e., the moduli of all multipliers are less than unity. From the condition $g(x, \ldots, x) = 0$, it follows that

$$\sum_{l=1}^{N} G^{(l)}(x) = 0\ . \tag{6.14}$$

Therefore, for $j=0$ the stability multiplier λ_0 associated with stability against the in-phase perturbation is

$$\lambda_0 = \prod_{t=0}^{p-1} f'(x(t))\ . \tag{6.15}$$

λ_0 is just the same as the stability multiplier of the uncoupled 1D map. While there is no coupling effect on λ_0, coupling generally affects other multipliers λ_j's of $j \neq 0$.

In the global coupling of the form (6.6), we first define $G(x)$ as

$$G(x) \equiv \frac{J}{N} u'(x)\ , \tag{6.16}$$

where $u'(x) = du/dx$. Then we have

$$\begin{aligned} G^{(2)}(x) &= \cdots = G^{(N)}(x) = G(x)\ , \\ G^{(1)}(x) &= (1-N)G(x)\ . \end{aligned} \tag{6.17}$$

Substituting these $G^{(l)}$'s into Eq. (6.13), we find that all stability multipliers λ_j's for nonzero j are the same:

$$\begin{aligned} \lambda_1 = \cdots = \lambda_{N-1} &= \prod_{t=0}^{p-1} [f'(x(t)) - NG(x(t))] \\ &= \prod_{t=0}^{p-1} [f'(x(t)) - Ju'(x(t))]\ . \end{aligned} \tag{6.18}$$

This implies that all modes with nonzero wave numbers have the same stability. Consequently there exist only two independent stability multipliers λ_0 and λ_1 $(=\lambda_2 = \cdots = \lambda_{N-1})$. Note also that all the stability multipliers (6.15) and (6.18) are independent of N.

As for the two coupled maps, let us choose $f(x) = 1 - Ax^2$ as the uncoupled 1D map. Then, in the $J-A$ parameter space, the stability diagram of in-phase orbits with period 2^n $(n=0,1,\ldots)$ in N globally coupled 1D maps for any $N>2$ is the same as that of two coupled maps because the two independent stability multipliers λ_0

and λ_1 for any N are the same as for the $N=2$ case [22]. In fact, the coupling parameter c of two coupled maps corresponds to $J/2$ and therefore the stability diagram can be obtained from the two-coupled map case by replacing c with $J/2$. Consequently the two parameter scaling factors γ_1 and γ_2 associated with scaling of the nonlinearity (A) and the coupling parameter (J), respectively, are the same regardless of N as those of two couple maps. Namely, the critical behavior of N globally coupled maps for $N>2$ is essentially the same as that of two coupled maps, in which case there exist three kinds of critical behaviors (for details of the $N=2$ case, refer to Ref. [10]).

C. Renormalization analysis

We follow the same procedure of the preceding sections for two coupled maps. The rescaling operator of Eq. (3.2) is αI, where I is now an $N\times N$ identity matrix. Applying the period-doubling renormalization operator $\mathcal{N}$ of Eq. (3.1) to the N-coupled map (6.3), we obtain the renormalized map T_1:

$$\begin{aligned} T_1:\ x_m(t+1)&=F_1(\sigma^{m-1}\mathbf{x}(t))\\ &=F_1(x_m(t),x_{m+1}(t),\ldots,x_{m-1}(t))\ ,\\ &\qquad m=1,\ldots,N\ . \end{aligned} \tag{6.19}$$

The renormalized function F_1 also consists of two parts, the uncoupled part f_1 and the coupling part g_1:

$$\begin{aligned} F_1(\mathbf{x})&=\alpha F\left[F\left[\frac{\mathbf{x}}{\alpha}\right],\ldots,F\left[\frac{\sigma^{N-1}\mathbf{x}}{\alpha}\right]\right]\\ &=f_1(x_1)+g_1(\mathbf{x})\ . \end{aligned} \tag{6.20}$$

Here the renormalized coupling function g_1 also obeys the condition (6.5), i.e., $g_1(x,\ldots,x)=0$ for any x. Therefore, the renormalized uncoupled function f_1 satisfies

$$f_1(x_1)=F_1(x_1,\ldots,x_1)=\alpha f\left[f\left[\frac{x_1}{\alpha}\right]\right]\ , \tag{6.21}$$

where the rescaling factor α is chosen to preserve the normalization condition $f_1(0)=1$, i.e., $\alpha=1/f(1)$. Subtracting f_1 from F_1, we have

$$\begin{aligned} g_1(\mathbf{x})=\alpha f\left[F\left[\frac{\mathbf{x}}{\alpha}\right]\right]+\alpha g\left[F\left[\frac{\mathbf{x}}{\alpha}\right],\ldots,F\left[\frac{\sigma^{N-1}\mathbf{x}}{\alpha}\right]\right]\\ -\alpha f\left[f\left[\frac{x_1}{\alpha}\right]\right]\ . \end{aligned} \tag{6.22}$$

Then, Eqs. (6.21) and (6.22) define a renormalization operator $\mathcal{R}$ of transforming a pair of functions (f,g):

$$\begin{bmatrix} f_1\\ g_1\end{bmatrix}=\mathcal{R}\begin{bmatrix} f\\ g\end{bmatrix}\ . \tag{6.23}$$

By successive iterations of $\mathcal{R}$, we obtain a recurrence equation:

$$\begin{bmatrix} f_{n+1}\\ g_{n+1}\end{bmatrix}=\mathcal{R}\begin{bmatrix} f_n\\ g_n\end{bmatrix}\ , \tag{6.24}$$

where the rescaling factor α is $1/f_n(1)$ and $f_n(g_n)$ is the uncoupled (coupling) part of the n times renormalized function F_n under the renormalization transformation $\mathcal{N}$.

Under iterations of $\mathcal{N}$ a critical map is attracted to a fixed map T^*:

$$\begin{aligned} T^*:\ x_m(t+1)&=F^*(\sigma^{m-1}\mathbf{x}(t))\\ &=f^*(x_m(t))\\ &\quad+g^*(x_m(t),x_{m+1}(t),\ldots,x_{m-1}(t))\ ,\\ &\qquad m=1,\ldots,N\ , \end{aligned} \tag{6.25}$$

where (f^*,g^*) is the fixed point of $\mathcal{R}$ with $\alpha=1/f^*(1)$. Since f^* is just the 1D fixed map, only the equation for the coupling fixed function g^* is left to be solved, which satisfies

$$\begin{aligned} g^*(\mathbf{x})=\alpha f^*\left[F^*\left[\frac{\mathbf{x}}{\alpha}\right]\right]\\ +\alpha g^*\left[F^*\left[\frac{\mathbf{x}}{\alpha}\right],\ldots,F^*\left[\frac{\sigma^{N-1}\mathbf{x}}{\alpha}\right]\right]\\ -\alpha f^*\left[f^*\left[\frac{x_1}{\alpha}\right]\right]\ . \end{aligned} \tag{6.26}$$

Like in the two-coupled map case, we construct a tractable recurrence equation for the reduced coupling function:

$$G^{(l)}(x)=\left.\frac{\partial g(\mathbf{x})}{\partial x_l}\right|_{x_1=\cdots=x_N=x}\ ,\quad l=1,\ldots,N\ . \tag{6.27}$$

That is, differentiating the recurrence equation (6.24) for g with respect to x_l ($l=1,\ldots,N$) and setting $x_1=\cdots=x_N=x$, we obtain

$$\begin{aligned} G_{n+1}^{(l)}(x)=f_n'\left[f_n\left[\frac{x}{\alpha}\right]\right]G^{(l)}\left[\frac{x}{\alpha}\right]\\ +G_n^{(l)}\left[f_n\left[\frac{x}{\alpha}\right]\right]f_n'\left[\frac{x}{\alpha}\right]\\ +\sum_{i=1}^{N}G_n^{(i)}\left[f_n\left[\frac{x}{\alpha}\right]\right]G_n^{(l-i+1)}\left[\frac{x}{\alpha}\right]\ . \end{aligned} \tag{6.28}$$

Note that these reduced coupling functions satisfy the sum rule of Eq. (6.14) and $G^{(l)}(x)=G^{(l+N)}(x)$ due to the periodic boundary condition.

In a global coupling case of the form (6.6), the initial reduced coupling functions $\{G^{(l)}(x)\}$ satisfy Eq. (6.17), i.e., there exists only one independent reduced coupling function $G(x)$. Then, it is easy to see that the successive images $\{G_n^{(l)}(x)\}$ of $\{G^{(l)}(x)\}$ under the transformation (6.28) also satisfy Eq. (6.17), i.e.,

$$\begin{aligned} G_n^{(2)}(x)&=\cdots=G_n^{(N)}(x)\equiv G_n(x)\ ,\\ G_n^{(1)}(x)&=(1-N)G_n(x)\ . \end{aligned} \tag{6.29}$$

Consequently, there remains only one recurrence equation for the independent reduced coupling function $G(x)$:

$$G_{n+1}(x)=\left[f_n'\left[f_n\left[\frac{x}{\alpha}\right]\right]-NG_n\left[f_n\left[\frac{x}{\alpha}\right]\right]\right]G_n\left[\frac{x}{\alpha}\right]+G_n\left[f_n\left[\frac{x}{\alpha}\right]\right]f_n'\left[\frac{x}{\alpha}\right]. \quad (6.30)$$

Then, together with the first row of Eq. (6.24), Eq. (6.30) defines a reduced renormalization operator $\tilde{\mathcal{R}}$ of transforming a pair of functions (f,G):

$$\begin{bmatrix} f_{n+1} \\ G_{n+1} \end{bmatrix}=\tilde{\mathcal{R}}\begin{bmatrix} f_n \\ g_n \end{bmatrix}. \quad (6.31)$$

Since the reduced renormalization transformation (6.31) holds for any globally coupled-map cases of $N\geq 2$, it can be regarded as a generalized version of Eq. (3.15) in the two-coupled-map case.

A pair of critical functions (f,G) is attracted to a pair of fixed functions (f^*,G^*) under iterations of the reduced renormalization operator $\tilde{\mathcal{R}}$, where f^* is the 1D fixed map and G^* satisfies

$$G^*(x)=\left[f^{*'}\left[f^*\left[\frac{x}{\alpha}\right]\right]-NG^*\left[f^*\left[\frac{x}{\alpha}\right]\right]\right]G^*\left[\frac{x}{\alpha}\right]+G^*\left[f^*\left[\frac{x}{\alpha}\right]\right]f^{*'}\left[\frac{x}{\alpha}\right]. \quad (6.32)$$

Here $G^*(x)$ is related with the reduced coupling fixed functions $\{G^{*(l)}\}$ in the same way as in Eq. (6.29), that is,

$$G^{*(2)}(x)=\cdots=G^{*(N)}(x)=G^*(x), \quad G^{*(1)}(x)=(1-N)G^*(x). \quad (6.33)$$

As in the $N=2$ case of Eqs. (3.19)–(3.21), we find three solutions for $G^*(x)$:

$$G^*(x)=0, \quad (6.34)$$

$$G^*(x)=\frac{1}{N}f^{*'}(x), \quad (6.35)$$

$$G^*(x)=\frac{1}{N}[f^{*'}(x)-1]. \quad (6.36)$$

For the same reason as for the two coupled maps [see Eq. (4.17)], the critical stability multipliers have the values of the stability multipliers of the fixed point of the fixed map T^*. From Eqs. (6.15) and (6.18), we obtain two independent critical stability multipliers λ_0^* and λ_1^*:

$$\lambda_0^*=f^{*'}(\hat{x}), \quad (6.37)$$

$$\lambda_1^*=f^{*'}(\hat{x})-NG^*(\hat{x}), \quad (6.38)$$

where $\hat{x}$ $(=0.5493\ldots)$ is the fixed point of the 1D fixed map $f^*(x)$. λ_0^* $(=-1.6011\ldots)$ is just the critical stability multiplier of the uncoupled 1D map and the other multipliers are the same as λ_1^*, i.e., $\lambda_1^*=\lambda_2^*=\cdots=\lambda_{N-1}^*$, as in Eq. (6.18). Substituting $G^*(x)$'s into Eq. (6.38), we have

$$\lambda_1^*=\begin{cases} \lambda_0^* \text{ for } G^*(x)=0 \\ 0 \text{ for } G^*(x)=\frac{1}{N}f^{*'}(x) \\ 1 \text{ for } G^*(x)=\frac{1}{N}[f^{*'}(x)-1]. \end{cases} \quad (6.39)$$

As noted earlier in Sec. VI B, λ_0^* and λ_1^* are independent of N, and hence they are just the critical stability multipliers in the $N=2$ case.

Now we examine how a pair of functions $(f^*(x_1)+h(x_1),\ g^*(\mathbf{x})+\varphi(\mathbf{x}))$ near a fixed point (f^*,g^*) evolves under the renormalization transformation $\mathcal{R}$. Linearizing $\mathcal{R}$ at the fixed point (f^*,g^*), we obtain an equation for the evolution of a pair of infinitesimal perturbations (h,φ):

$$\begin{bmatrix} h_{n+1} \\ \varphi_{n+1} \end{bmatrix}=\mathcal{L}\begin{bmatrix} h_n \\ \varphi_n \end{bmatrix}=\begin{bmatrix} \mathcal{L}_1 & 0 \\ \mathcal{L}_3 & \mathcal{L}_2 \end{bmatrix}\begin{bmatrix} h_n \\ \varphi_n \end{bmatrix}, \quad (6.40)$$

where

$$h_{n+1}(x_1)=[\mathcal{L}_1h_n](x_1)=\alpha f^{*'}\left[f^*\left[\frac{x_1}{\alpha}\right]\right]h_n\left[\frac{x_1}{\alpha}\right]+\alpha h_n\left[f^*\left[\frac{x_1}{\alpha}\right]\right], \quad (6.41)$$

$$\varphi_{n+1}(\mathbf{x})=[\mathcal{L}_2\varphi_n](\mathbf{x})+[\mathcal{L}_3h_n](x_1), \quad (6.42)$$

$$[\mathcal{L}_2\varphi_n](\mathbf{x})=\sum_{i=1}^{N}\alpha F_i^*\left[F^*\left[\frac{\mathbf{x}}{\alpha}\right],\ldots,F^*\left[\frac{\sigma^{N-1}\mathbf{x}}{\alpha}\right]\right]\times\varphi_n\left[\frac{\sigma^{i-1}\mathbf{x}}{\alpha}\right]+\alpha\varphi_n\left[F^*\left[\frac{\mathbf{x}}{\alpha}\right],\ldots,F^*\left[\frac{\sigma^{N-1}\mathbf{x}}{\alpha}\right]\right], \quad (6.43)$$

$$[\mathcal{L}_3h_n](x_1)=\sum_{i=1}^{N}\alpha F_i^*\left[F^*\left[\frac{\mathbf{x}}{\alpha}\right],\ldots,F^*\left[\frac{\sigma^{N-1}\mathbf{x}}{\alpha}\right]\right]\times h_n(x_i)+\alpha h_n\left[F^*\left[\frac{\mathbf{x}}{\alpha}\right]\right]-[\mathcal{L}_1h_n](x_1). \quad (6.44)$$

Here $F^*(\mathbf{x})=f^*(x_1)+g^*(\mathbf{x})$, and the subscript i $(i=1,\ldots,N)$ of F_i^* denotes the partial derivative with respect to the ith argument. The reducibility of $\mathcal{L}$ into a semiblock form, following the same arguments as for the two coupled maps, implies that the coupling eigenvalues of the present interest can be obtained only by solving

$$\lambda\varphi(\mathbf{x})=\mathcal{L}_2\varphi(\mathbf{x}). \quad (6.45)$$

We introduce reduced coupling eigenperturbations by

$$\Phi^{(l)}(x)=\left.\frac{\partial\varphi(\mathbf{x})}{\partial x_l}\right|_{x_1=\cdots=x_N=x}, \quad l=1,\ldots,N. \quad (6.46)$$

Then, differentiating the eigenvalue equation (6.45) with respect to x_l and setting $x_1=\cdots=x_N=x$, we have a set of N equations:

$$\lambda\Phi^{(l)}(x)=f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]\Phi^{(l)}\left[\frac{x}{\alpha}\right]+\Phi^{(l)}\left[f^*\left[\frac{x}{\alpha}\right]\right]f^{*\prime}\left[\frac{x}{\alpha}\right]$$
$$+\sum_{i=1}^{N}\left[G^{*(i)}\left[f^*\left[\frac{x}{\alpha}\right]\right]\Phi^{(l-i+1)}\left[\frac{x}{\alpha}\right]+\Phi^{(i)}\left[f^*\left[\frac{x}{\alpha}\right]\right]G^{*(l-i+1)}\left[\frac{x}{\alpha}\right]\right]. \quad (6.47)$$

In a global coupling case of the form (6.6), the reduced coupling eigenperturbations $\{\Phi^{(l)}\}$ satisfy

$$\Phi^{(2)}(x)=\cdots=\Phi^{(N)}(x)\equiv\Phi(x)\ ,$$
$$\Phi^{(l)}(x)=(1-N)\Phi(x)\ , \quad (6.48)$$

i.e., there exists only one independent reduced coupling eigenperturbation $\Phi(x)$. Substituting the $\Phi^{(l)}$'s and $G^{*(l)}$'s into Eq. (6.47), we obtain an eigenvalue equation for $\Phi(x)$:

$$\lambda\Phi(x)=\left[f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]-NG^*\left[f^*\left[\frac{x}{\alpha}\right]\right]\right]\Phi\left[\frac{x}{\alpha}\right]$$
$$+\left[f^{*\prime}\left[\frac{x}{\alpha}\right]-NG^*\left[\frac{x}{\alpha}\right]\right]\Phi\left[f^*\left[\frac{x}{\alpha}\right]\right]. \quad (6.49)$$

The reduced eigenvalue equation (6.49) can also be obtained as follows. Consider an infinitesimal perturbation $(0,\Phi)$ to a fixed point (f^*,G^*) of the reduced renormalization operator $\tilde{\mathcal{R}}$ of Eq. (6.31). Linearizing $\tilde{\mathcal{R}}$ at the fixed point (f^*,G^*), we obtain an equation for the evolution of Φ:

$$\Phi_{n+1}(x)=[\tilde{\mathcal{L}}_2\Phi_n](x)$$
$$=\left[f^{*\prime}\left[f^*\left[\frac{x}{\alpha}\right]\right]-NG^*\left[f^*\left[\frac{x}{\alpha}\right]\right]\right]\Phi_n\left[\frac{x}{\alpha}\right]$$
$$+\left[f^{*\prime}\left[\frac{x}{\alpha}\right]-NG^*\left[\frac{x}{\alpha}\right]\right]\Phi_n\left[f^*\left[\frac{x}{\alpha}\right]\right]. \quad (6.50)$$

The eigenvalue equation of the reduced linearized operator $\tilde{\mathcal{L}}_2$ is just the one of Eq. (6.49), which is the same as Eq. (4.14) of the two-coupled-map case except for the factor N. Following the same procedure, therefore, the relevant coupling eigenvalues λ in N-globally-coupled-map case for $N>2$ are obtained as follows.

(1) $G^*(x)=0$:

$$\lambda=\begin{cases}\alpha\ \text{(linear coupling case)}\\ 2\ \text{(nonlinear coupling case)}\ .\end{cases} \quad (6.51)$$

(2) $G^*(x)=(1/N)f^{*\prime}(x)$: no relevant coupling eigenvalue.

(3) $G^*(x)=(1/N)[f^{*\prime}(x)-1]$: $\lambda=2$.

VII. SUMMARY

The critical behavior of period doubling in two coupled maps is studied by a renormalization method. We find three kinds of fixed maps, each of which has a common relevant eigenvalue associated with scaling of the nonlinearity parameter. But the relevant coupling eigenvalues associated with scaling of the coupling parameter vary depending on the kind of the fixed maps. Their values agree well with the numerical values of the coupling parameter scaling factor. We also extend the renormalization results of two coupled maps to many-globally-coupled-map case.

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APPENDIX A: COUPLING EIGENVALUES ASSOCIATED WITH COORDINATE CHANGES

In this appendix, we obtain coupling eigenvalues (4.9) associated with smooth coordinate changes.

Consider an infinitesimal coordinate change $S=I+\epsilon U$ (ϵ small). Here I is the identity transformation and U is of the form

$$U(\mathbf{x})=(U_1(\mathbf{x}),U_2(\mathbf{x}))\ , \quad (A1)$$

where $\mathbf{x}=(x,y)$, $U_1(x,y)=(y-x)\psi(x,y)$, and $U_2(x,y)=U_1(y,x)$. This coordinate change generates an infinitesimal coupling perturbation $\epsilon\delta T$ to a fixed map T^* $[x_{i+1}=F^*(x_i,y_i),\ y_{i+1}=F^*(y_i,x_i)]$;

$$\delta T(\mathbf{x})=(\varphi(x,y),\varphi(y,x))\ , \quad (A2)$$

where

$$\varphi(x,y)=U_1(F^*(x,y),F^*(y,x))-U_1(x,y)F_1^*(x,y)-U_2(x,y)F_2^*(x,y)$$
$$=[F^*(y,x)-F^*(x,y)]\psi(F^*(x,y),F^*(y,x))-(y-x)F_1^*(x,y)\psi(x,y)-(x-y)F_2^*(x,y)\psi(y,x)\ . \quad (A3)$$

Here the subscript i $(i=1,2)$ of F^* denotes the partial derivative with respect to the ith argument. The perturbation $\varphi(x,y)$ is transformed into φ_1 under the linearized renormalization transformation $\mathcal{L}_2$ of Eq. (4.4):

$$\varphi_1(x,y)=[\mathcal{L}_2\varphi](x,y) \tag{A4}$$

$$=[F^*(y,x)-F^*(x,y)]\psi\left[\frac{F^*(x,y)}{\alpha},\frac{F^*(y,x)}{\alpha}\right] -(y-x)F_1^*(x,y)\psi\left[\frac{x}{\alpha},\frac{y}{\alpha}\right] -(x-y)F_2^*(x,y)\psi\left[\frac{y}{\alpha},\frac{x}{\alpha}\right] . \tag{A5}$$

Therefore, φ becomes an eigenfunction of $\mathcal{L}_2$ with eigenvalue $\lambda=\alpha^{-(l+m)}$ when $\psi(x,y)=\psi_{l,m}(x,y)\equiv x^l y^m$ $(l,m\geq 0)$. When $l+m>0$, all eigenvalues are irrelevant. When $l=m=0$, we have the marginal eigenvalue $\lambda=1$, which corresponds to the scale change of Y $[=(x-y)/2]$; see Eq. (5.8). In the case of the fixed map (5.27), $\varphi(x,y)$ corresponding to the marginal eigenvalue becomes identically zero, unlike the other two cases of fixed maps (5.2) and (5.15). This is because the fixed map (5.27) is invariant under the coordinate change (5.8).

The reduced coupling eigenfunction Φ of the above coupling eigenfunction is

$$\Phi(x)=[f^{*'}(x)-2G^*(x)][f^{*l+m}(x)-x^{l+m}] . \tag{A6}$$

This is just the eigenfunction of the reduced eigenvalue equation (4.14) with eigenvalue $\lambda=\alpha^{-(l+m)}$. Note that the marginal eigenvalue case cannot be obtained from the reduced eigenvalue equation because $\Phi(x)=0$ for $l+m=0$.

APPENDIX B: METHOD TO OBTAIN THE QUADRATIC FIXED MAP

In the following, we outline a method to obtain a quadratic approximant (QA) of a fixed map, i.e., to determine a fixed map up to its quadratic terms in the system of coordinates X and Y of Eq. (5.3).

Expressing the critical map T_c of Eq. (3.12) in terms of X and Y, we have

$$\begin{aligned}X_{i+1}&=R(X_i,Y_i)\\&=\tfrac{1}{2}[f_c(X_i+Y_i)+f_c(X_i-Y_i)]\\&\quad+\tfrac{1}{2}[g_c(X_i+Y_i,X_i-Y_i)+g_c(X_i-Y_i,X_i+Y_i)] ,\\Y_{i+1}&=S(X_i,Y_i)\\&=\tfrac{1}{2}[f_c(X_i+Y_i)-f_c(X_i-Y_i)]\\&\quad+\tfrac{1}{2}[g_c(X_i+Y_i,X_i-Y_i)-g_c(X_i-Y_i,X_i+Y_i)] .\end{aligned} \tag{B1}$$

Note that R is even and S is odd in Y.

The nth image T_n of T_c under $\mathcal{N}$ is

$$T_n=\mathcal{N}^n(T_c)=B_nT_{n-1}^2B_n^{-1}=\Lambda_nT_c^{2^n}\Lambda_n^{-1} , \tag{B2}$$

where

$$B_n=\begin{bmatrix}\alpha_n & 0\\ 0 & \alpha_n\end{bmatrix} , \quad \Lambda_n=\begin{bmatrix}\beta_n & 0\\ 0 & \beta_n\end{bmatrix} , \tag{B3}$$

$$\alpha_n=\frac{1}{f_{n-1}(1)} , \quad \beta_n=\prod_{i=1}^{n}\alpha_i=\frac{1}{f_c^{2^n}(0)} . \tag{B4}$$

Here $f_0=f_c$ and f_n is the n-times renormalized map, i.e., $f_n(x)=\alpha_nf_{n-1}^2(x/\alpha_n)$. As n goes to the infinity, T_n converges to a fixed map T^*: $\lim_{n\to\infty}\mathcal{N}^n(T_c)=T^*$.

T_n of Eq. (B2) consists of iterating T_c 2^n times and then rescaling. Therefore, to obtain a QA of T_n, the procedure is divided into two steps: (1) Obtain a QA of $T_c^{2^n}$. (2) Rescale the coordinates X and Y by $f_c^{2^n}(0)$.

We first determine $T_c^{2^n}$ up to quadratic terms. Consider the mth iterate of T_c, i.e., T_c^m; hereafter, we will use $m=2^n$. That is, denoting as (X_m, Y_m) the mth image of (X,Y) under T_c,

$$X_m\equiv R^{(m)}(X,Y)=R(X_{m-1},Y_{m-1}) , \tag{B5}$$

$$Y_m\equiv S^{(m)}(X,Y)=S(X_{m-1},Y_{m-1}) , \tag{B6}$$

where $(X_0,Y_0)=(X,Y)$, and the functions $R^{(m)}$ and $S^{(m)}$ are even and odd in Y, respectively.

To proceed, we will Taylor expand the functions in Eqs. (B5) and (B6) about the origin (0,0). For brevity, we will denote the origin simply as 0, and $(\hat{X}_{m-1}, \hat{Y}_{m-1})$, which is the $(m-1)$th image of the origin under T_c, as $\hat{Z}_{m-1}$. First, expand $R^{(m)}$ and $S^{(m)}$ about the origin to obtain

$$X_m=R^{(m)}(0)+R_1^{(m)}(0)X+R_2^{(m)}(0)Y+\tfrac{1}{2}R_{1,1}^{(m)}(0)X^2 +R_{1,2}^{(m)}(0)XY+\tfrac{1}{2}R_{2,2}^{(m)}(0)Y^2+\cdots , \tag{B7}$$

$$Y_m=S^{(m)}(0)+S_1^{(m)}(0)X+S_2^{(m)}(0)Y+\tfrac{1}{2}S_{1,1}^{(m)}(0)X^2 +S_{1,2}^{(m)}(0)XY+\tfrac{1}{2}S_{2,2}^{(m)}(0)Y^2+\cdots , \tag{B8}$$

where the subscript i $(i=1,2)$ denotes the partial derivative with respect to the ith argument. Next, a Taylor expansion of the second expressions in Eqs. (B5) and (B6), i.e., $R(X_{m-1},Y_{m-1})$ and $S(X_{m-1},Y_{m-1})$, about the origin gives

$$\begin{aligned}X_m&=R(\hat{Z}_{m-1})+R_1(\hat{Z}_{m-1})R_1^{(m-1)}(0)X\\&\quad+R_2(\hat{Z}_{m-1})S_1^{(m-1)}(0)X+R_1(\hat{Z}_{m-1})R_2^{(m-1)}(0)Y\\&\quad+R_2(\hat{Z}_{m-1})S_2^{(m-1)}(0)Y+\cdots ,\end{aligned} \tag{B9}$$

$$\begin{aligned}Y_m&=S(\hat{Z}_{m-1})+S_1(\hat{Z}_m-1)R_1^{(m-1)}(0)X\\&\quad+S_2(\hat{Z}_{m-1})S_1^{(m-1)}(0)X+S_1(\hat{Z}_{m-1})R_2^{(m-1)}(0)Y\\&\quad+S_2(\hat{Z}_{m-1})S_2^{(m-1)}(0)Y+\cdots .\end{aligned} \tag{B10}$$

It can easily be shown that the constant terms are $R^{(m)}(0,0)=f_c^m(0)$ and $S^{(m)}(0,0)=0$. By comparing the coefficients of two expansions, the coefficients of linear and quadratic terms can also be obtained recursively, which is represented in a matrix form as follows:

$$\underline{L}_m = \begin{bmatrix} R_1 & R_2 \\ S_1 & S_2 \end{bmatrix}_{\hat{z}_{m-1}} \underline{L}_{m-1} , \qquad \text{(B11)}$$

$$\underline{Q}_m = \begin{bmatrix} R_1 & R_2 \\ S_1 & S_2 \end{bmatrix}_{\hat{z}_{m-1}} \underline{Q}_{m-1} + \begin{bmatrix} R_{1,1} & R_{1,2} & R_{2,2} \\ S_{1,1} & S_{1,2} & S_{2,2} \end{bmatrix}_{\hat{z}_{m-1}} \underline{M}_{m-1} , \qquad \text{(B12)}$$

where

$$\underline{L}_m \equiv \begin{bmatrix} R_1^{(m)} & R_2^{(m)} \\ S_1^{(m)} & S_2^{(m)} \end{bmatrix}_0 , \quad \underline{Q}_m \equiv \begin{bmatrix} R_{1,1}^{(m)} & R_{1,2}^{(m)} & R_{2,2}^{(m)} \\ S_{1,1}^{(m)} & S_{1,2}^{(m)} & S_{2,2}^{(m)} \end{bmatrix}_0 , \qquad \text{(B13)}$$

$$\underline{M}_m \equiv \begin{bmatrix} R_1^{(m)2} & R_1^{(m)}R_2^{(m)} & R_2^{(m)2} \\ 2R_1^{(m)}S_1^{(m)} & R_1^{(m)}S_2^{(m)}+R_2^{(m)}S_1^{(m)} & 2R_2^{(m)}S_2^{(m)} \\ S_1^{(m)2} & S_1^{(m)}S_2^{(m)} & S_2^{(m)2} \end{bmatrix}_0 , \qquad \text{(B14)}$$

where the subscript i $(i=1,2)$ denotes the partial derivative with respect to the ith argument. Note that $\underline{M}_m$ can be determined from the elements of $\underline{L}_m$. We thus have the QA of T_c^m:

$$\begin{bmatrix} X_m \\ Y_m \end{bmatrix} = \begin{bmatrix} f_c^m(0) \\ 0 \end{bmatrix} + \underline{L}_m \begin{bmatrix} X \\ Y \end{bmatrix} + \frac{1}{2}\underline{Q}_m \begin{bmatrix} X^2 \\ 2XY \\ Y^2 \end{bmatrix} . \qquad \text{(B15)}$$

Rescaling the coordinates X and Y by β_n^{-1} $[=f_c^{2^n}(0)]$, we obtain the QA of T_n as the following form:

$$\begin{bmatrix} X_{i+1} \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underline{L}_{2^n} \begin{bmatrix} X_i \\ Y_i \end{bmatrix} + \frac{1}{2\beta_n}\underline{Q}_{2^n} \begin{bmatrix} X_i^2 \\ 2X_iY_i \\ Y_i^2 \end{bmatrix} . \qquad \text{(B16)}$$

The QA of T_n converges to that of T^* as $n \to \infty$.

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